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by

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AERODYNAMIC THEORY OF THE ANNULAR JET[+]

By

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ABSTRACT

A theory is presented to describe the flow mechanics of air jets issuing from annular openings. Such jets are termed "annular jets". Two configurations are studied:

1. Annular nozzle far from the ground. The jet takes the form of a hollow tulip attached to the orifice of the nozzle and the thrust of the annular nozzle is smaller than that of the corresponding circular nozzle, since it is reduced by the negative pressure on the nozzle base plate resulting from the "jet pump" action of the annular jet. The base pressure is calculated theoretically for a "two dimensional" viscous laminar or turbulent, incompressible or compressible jet, using a "dividing streamline" concept. A solution of the three-dimensional problem is given in principle; this solution makes use of a Mangler transformation. Calculated base pressures are found to agree fairly well with those found experimentally.

2. Annular nozzle close to the ground. The annular jet, initially vertical, spreads out radially in the horizontal direction, thus creating a region of high pressure air under the nozzle base plate. The thrust of the annular nozzle is now larger than that of the corresponding circular nozzle since it is the sum of the direct jet thrust and of the pressure reaction between the ground and the nozzle base. The variation of the thrust augmentation with the distance from the ground can be explained in terms of simple momentum considerations and good agreement with experimental results is found. The velocity distribution within the curved jet can only be found if viscous effects are introduced;a solution to the viscous problem is sketched in the two-dimensional case. Finally, existing experimental data are briefly reviewed and the parameters which make possible a correlation of these data are pointed out.

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INTRODUCTION

The flow properties, i.e., the spatial distribution of velocity, pressure, temperature and density, of a viscous jet issuing from two-dimensional and circular openings have been studied extensively both from the theoretical and from the experimental point of view and considerable success has been achieved in describing the flow analytically (See, Reference 1, for example). Little attention has been given, however, to the analytic formulation of the flow properties of jets issuing from non-simple orifices. In particular, there is very little information available concerning the flow properties of jets issuing from annular openings. Such jets are termed "annular jets" and in recent months much interest in their behavior has been expressed in various quarters.

Interest in annular jets originates in their anomalous behavior when in close proximity to a boundary normal to the axis of the jet. Unlike a circular jet, which exhibits a thrust loss under such conditions due to unfavorable interference by the normal boundary, annular jets experience a favorable ground interference resulting in a pressure rise within the cavity formed by the jet sheets, the nozzle centerbody and the boundary. The excess pressure Δp acting across the jet sheets causes them to curve outward symmetrically. The distribution of this excess pressure across the nozzle centerbody of area S_b gives rise to a term $S_b \Delta p$, which augments the thrust due to the momentum flux of the jet. The magnitude of the thrust augmentation increases with proximity of the nozzle configuration to the normal boundary. This feature seems to offer the possibility of constructing a hovering machine powered with an engine yielding a delivered thrust much smaller than the gross weight of the fully loaded machine. The hovering capability of such a machine would undoubtedly be limited to rather close proximity to the ground. However, the very fact that the machine is capable of raising itself above the surface of the earth and travel on a cushion of air without the use of wheels, tracks, hulls or hydrofoils permits its adaptation to a rather wide number of uses. It is not the purpose of this paper to enter into a detailed description of the various types of "ground effect machines" proposed or under development (See for this References 2 and 3), but rather to discuss the aerodynamic theory of the annular jet, since the operation of these machines depends upon the annular jet. It is thought that mention of such vehicles is in order since an incentive for a theoretical study of the annular jet is thus provided. It is believed that costly development programs can be shortened if a fundamental understanding of the principles underlying the operation of ground effect machines is obtained early in the game.

Another characteristic feature of annular jets is that, far from the ground, their thrust is smaller than that of the corresponding circular nozzle (nozzle of same exhaust area and momentum flux). The thrust is reduced as a result of the action of a negative pressure on the nozzle base plate resulting from the "jet pump" action of the annular jet. This is an undesirable effect and it is of interest to study it, if only to try to find ways to alleviate it.

Both annular jets far from the ground and close to the ground are summarily discussed in this paper. For more details, the reader is referred to the bibliography at the end of the paper, especially References 4, 5 and 6. The state of the theory is far from being satisfactory yet and in most cases no experimental data are available to check it. It is, however, hoped that it

will lay the groundwork for and stimulate interest in the *quantitative* understanding of annular jets.

BASE PRESSURE OF ANNULAR JETS

Experimental Data

There exists very few data concerning the "base pressure" of an annular nozzle far from the ground, all of which are contained in Reference 4. The nozzle which was tested was an axially symmetric nozzle. No results concerning the "two-dimensional" analogue of the annular jet (i.e., two infinite, plane, parallel jets close to each other) are available. The three dimensional results were obtained for pressure ratios P_n/p_o between 1.2 and 2.7. In Figure 1 a plot of $F_t/S_j p_j$ vs p_n/p_o is shown, where F_t represents the total thrust of an annular or of a circular nozzle outside the ground effect, S_j is the nozzle area and p_j the jet total pressure (gage). It is found that the measured thrust of circular nozzles (E and F) is lower than that calculated from isentropic flow relations (upper curve) by about ten percent. It is found further that the thrust of an annular nozzle is less than that of the corresponding circular nozzle by an amount which corresponds to the "base pressure". The corresponding thrust loss, F_L, is plotted separately against pressure ratio in Figure 2 (where S_t represents the total nozzle area). It can be seen from this figure that there is a scatter of the experimental points, but that a curve can be fitted to represent most of them.

Typical photographs of the jet shape issuing from the annular nozzle indicate that the annular jet stream takes the form of a hollow tulip and converges on the axis of revolution of the nozzle, as shown schematically in Figures 3 and 11 leaving a dead-air region inside the cavity formed by the jet sheets and the nozzle.

From Figure 2 and the data of Reference 4, one can calculate the ratio p_c/p' of the pressure within the cavity to the static pressure outside and plot it against nozzle pressure ratio P_n/p_o. This is done in Figure 8. It can be seen that the effect of the base pressure is to reduce the thrust of the annular jet between five and ten percent.

Relation of Annular Jet to Blunt-Body Base Pressure Problem

In the same way in which a negative pressure appears over the centerbody of an annular nozzle, it is well known that the pressure at the rear of a blunt-base projectile, either in subsonic or in supersonic flight, is less than atmospheric, resulting in a drag. Therefore, a review of the methods of analysis used for evaluating the base pressure of missiles was made in order to see if they could be extended to the annular jet (References 7 to 15). The methods used in base pressure problems can be classified as empirical (Reference 7), semi-empirical (References 8 and 9) and theoretical (References 10 to 15). The use of empirical or semi-empirical methods had to be ruled out for the annular jet on account of the nearly complete lack of experimental data noted previously. After some study, it was found that the theoretical scheme used by Chapman, Korst and Carrière in References 11 to 15 for calculating missile base pressures could be applied to the case of the annular jet. The scheme is only applied by these authors to a two-dimensional flow.

Conceptual Scheme, "Two-Dimensional" Annular Nozzle

As noted previously, a "two-dimensional" annular nozzle consists of two infinite parallel slot jets close to each other, such as represented schematically in Figure 3. Inspection of Figure 3 shows that xx' is a plane of symmetry; thus, it is necessary to study the flow of only one of the jets under the influence of the boundaries shown in the figure, where xx' has been replaced by a hypothetical solid boundary or "reflection plane".

Ignoring the effect of boundary layers developed in the interior of the nozzle, if it is assumed that the jet emerges from the nozzle with a uniform velocity distribution, it is clear that viscous mixing at the jet boundaries will occur resulting in flow entrainment as shown in Figure 3, including a rotational motion inside the dead-air region. As a result, the initially *uniform velocity distribution decays rapidly until a typical bell-shaped profile* is established a few nozzle widths downstream of the nozzle. Downstream the width of the mixing region increases and the velocity distribution decays in such a manner that the total momentum flux across any plane normal to the axis of the jet remains constant. It is clear that viscous expansion of the jet will eventually seal off the cavity interior from the external atmosphere. Once this happens, entrainment of air from the cavity interior will lower the pressure and the resulting pressure difference across the jet will cause it to curve inward, eventually resulting in one of the streamlines stagnating at point R. The distribution of velocities along a line normal to the jet streamlines is as shown in Figure 3. The zone EJI is the mixing zone where the exchange of momentum between the jet, the surrounding air, and the dead-air region takes place.

In the calculation of dead-air pressure, the essential mechanism involved is assumed to be a balance between the mass flow entrained by viscous exchange of momentum in the mixing zone and the mass flow reversed back into the cavity interior in the neighborhood of point R.

Let BR be the dividing line separating the jet particles which flow downstream from those which are recirculated in the dead-air region. Streamlines whose total pressure is greater than that of BR pass downstream, while those whose total pressure is less than that of BR are recirculated in the cavity interior. Inasmuch as no mass flow is added to or removed from the interior of the cavity, it is clear that the dividing streamline must originate at the lower edge of the nozzle, as shown in Figure 3.

Consider now a particle traveling along the streamline BR of Figure 3. The pressure at B is P_1, while the pressure at R is P_2, the terminal static pressure, which is larger than P_1. The particle has a velocity at B equal to the initial velocity at the edge of the jet. As the particle travels along BR, it exchanges part of its momentum with the neighboring particles in the cavity, so that when it reaches the vicinity of point R its velocity has decreased, its pressure remaining P_1. The particle then encounters an adverse pressure gradient corresponding to the change of pressure from P_1 to P_2. Under the influence of this gradient, the velocity of the particle becomes zero at R.

For purpose of analysis, the flow field is divided into two regions as illustrated in Figure 4. Region I is regarded as a region wherein viscous effects predominate and in which the static pressure of the jet is assumed constant and

numerically equal to the cavity pressure P_c ($P_1 = P_c$). The total pressure along any streamline, e.g., the dividing streamline, decreases with distance due to the dissipation of energy by viscosity. Region II is regarded as a reattachment zone wherein the flow experiences a compression from the static pressure P_c to the static pressure at the end of the reattachment zone, p' ($p' = P_2$ of Figure 3), which is taken to be that of the ambient atmosphere. It is assumed that the viscous dissipation of energy in this region is small, so that the variation of total pressure along the dividing streamline in this region is negligible. Hence, the total pressure along this streamline may be regarded as a constant, equal to its value at the terminus of region I. With the aid of these assumptions, the cavity pressure may be calculated in a manner similar to Chapman's or Korst's treatment of the missile base pressure problem, *References 12 and 14*. Designating P_c as the pressure in the dead-air region of the cavity and $\overline{M}_d$ as the Mach number of the dividing streamline at terminus of region I, the cavity pressure is determined by equating the total pressure along the dividing streamline at the terminus of region I, $\overline{p}$, to the static pressure at the end of the reattachment zone, p'.

$$\overline{p} = p'$$

But:

$$\overline{p} = p\left[1 + \frac{\gamma - 1}{2}\,\overline{M}_d^{\,2}\right]^{\gamma/\gamma - 1}$$

and

$$p = P_c$$

Thus:

$$P_c = \frac{p'}{\left[1 + \frac{\gamma - 1}{2}\,\overline{M}_d^{\,2}\right]^{\gamma/\gamma - 1}}$$

The problem reduces itself to one of determining the velocity components on the dividing streamline at the start of the recompression zone.

The velocity components of the slot jet under consideration can be calculated rather simply if one assumes that the curvature of the jet shown in Figure 3 will not alter appreciably its velocity distribution. In other terms, one assumes that one can calculate a velocity distribution for the configuration of *Figure 4 (straight jet)and apply it to the curved jet of Figure 3, or that the mixing* of the jet is not greatly affected by its curvature. This may not be a good assumption, but a similar one is made by Chapman (Reference 12).

Application to a Laminar, Incompressible, Two-Dimensional Annular Nozzle

When one has a slot jet, one takes the center line of the jet as x-axis, a y-axis perpendicular to it, and the origin at the slot (Figure 5). For a curved jet, one still assumes that there exists a "center line" of the jet which gives the general shape of the jet, at least for a thin jet, and one takes a curvilinear set of coordinates, x being measured parallel to this center line

and y normal to it (Figure 6).

In both cases, u and v represent the velocity components in the x and y direction, respectively.

Navier-Stokes equations for a thin jet are of a "boundary layer nature", meaning that the region of space in which a solution is being sought does not extend far in a transverse direction as compared with the main direction of flow and that the transverse gradients are large. In what follows, the boundary-layer approximations will be applied in all cases.

In particular, the equations of motion of a two-dimensional laminar incompressible jet mixing with a fluid at rest are:

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \nu\dfrac{\partial^2 u}{\partial y^2} \\[2ex] \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

The boundary conditions of this problem are:

$$y = 0 : v = 0, \quad \frac{\partial u}{\partial y} = 0$$

$$y = \pm\infty : u = 0$$

In addition, the total momentum flux across any cross-section of the flow field normal to the jet axis is constant and equal to the momentum flux per unit width, J, discharged at the origin.

The solution to these equations was found by Schlichting and Bickley (*References 16 and 17*) and *reads as follows:*

defining a stream function Ψ such as

$$u = \frac{\partial \Psi}{\partial y} \qquad v = -\frac{\partial \Psi}{\partial x}$$

and assuming that the velocity profiles are similar, such as:

$$\eta = y\,x^{-2/3}$$

one finds:

$$\Psi = 1.6510\,(K\nu x)^{1/3}\tanh\xi$$

$$u = \frac{0.4543\left(\frac{K^2}{\nu}\right)^{1/3}}{x^{1/3}}\,(1 - \tanh^2\xi)$$

$$v = \frac{0.5503(K\nu)^{1/3}}{x^{2/3}}\left(2\xi\ \mathrm{sech}^2\xi - \tanh\xi\right)$$

and $$\xi = 0.2752\left(\frac{K}{\nu^2}\right)^{1/3}\frac{y}{x^{2/3}}$$

where: $K = j/\rho$ and $\nu = \mu/\rho$

μ is the viscosity and ρ the density.

From the equations, one must find the value of the velocity components on the dividing streamline at the start of the recompression zone.(end of region I of Figure 4). Since the entire mass flux, m, (per unit width of the jet) is contained between the two dividing streamlines, it is clear that one has at every cross-section of the jet:

$$2\rho\int_0^{y_d} u\,dy = m$$

where y_d is the ordinate of the dividing streamline. Designating by Ψ_d the value of Ψ corresponding to y_d, it follows that:

$$\int_0^{\Psi_d}\frac{d\Psi}{dy}\,dy = \frac{1}{2}\left(\frac{m}{\rho}\right)_0 = \frac{1}{2}m$$

Integrating, there comes:

$$\Psi_d = \frac{1}{2}m$$

On the dividing streamline, one has:

$$\Psi_d = 1.6510\,(K\nu x)^{1/3}\tanh\xi_d = \frac{1}{2}m$$

hence:

$$\tanh\xi_d = \frac{0.3028\,\frac{m}{(K\nu)^{1/3}}}{x^{1/3}}$$

$$u_d = \frac{0.4543\left(\frac{K^2}{\nu}\right)^{1/3}}{x^{1/3}}\left(1 - \tanh^2\xi_d\right)$$

$$v_d = \frac{0.5503(K\nu)^{1/3}}{x^{2/3}}\left(2\xi_d\ \mathrm{sech}^2\xi_d - \tanh\xi_d\right)$$

Therefore u_d and v_d are expressed as a function of x alone.

A numerical example was calculated, corresponding to nozzle A of Reference 4, for a pressure ratio of 1.55. One sees first that v_d is very small compared to u_d and can be disregarded. Let U be the velocity on the axis of the jet. U and u_d are plotted against x in Figure 7. It can be seen that the velocity U is infinite for x = 0, as well as u_d (which becomes $-\infty$). This infinity comes from disregarding the diffusion terms in Navier-Stokes equation. One makes therefore a translation of the y-axis:

$$X = x - x_0$$

such that, at $x = x_0$, U will be equal to the initial velocity of the jet. In the present case, one finds x_0 = 664 ft. This may look like a ridiculous result, but it only expresses the fact that in the laminar equations the velocity is not damped very quickly. One must therefore displace the y axis 664 ft. towards the right in Figure 7. One notices the remarkable fact that the velocity along the dividing streamline is approximately constant. Note that in Chapman's case (Reference 12) the velocity along the dividing streamline was also approximately constant. This fact permits a great simplification of the computational effort, since one does not have to determine the location of the start of the recompression zone. From Figure 7, one finds u_d = 505 ft. Hence $\bar{M}_d = 0.453$, $p_c/p' = 0.869$.

This "two-dimensional" theoretical value of the base pressure is plotted in Figure 8. It is of interest to note that the thrust loss predicated theoretically is about 13 percent while three-dimensional experimental data indicate a thrust loss of about 4 percent, therefore, a ratio of 3 to 1 between two-dimensional and three-dimensional values. This is precisely the ratio between two-and three-dimensional values found empirically by Hoerner in Reference 7.

It is of interest for the above jet to plot the locus of the ordinate of the dividing streamline y_d against x, as well as that of the "edge" of the jet, which is defined such as $u/U = 0.001$. Both are plotted in Figure 9. It can be seen that the dividing streamlines diverge only slightly, i.e., the jet retains its identity and remains essentially thin.

Laminar, Compressible, Two-Dimensional Annular Nozzle

The equations governing the two-dimensional laminar compressible jet are:

$$\begin{cases} \rho u \dfrac{\partial u}{\partial x} + \rho v \dfrac{\partial u}{\partial y} = \dfrac{\partial}{\partial y}\left(\mu \dfrac{\partial u}{\partial y}\right) \\ \dfrac{\partial(\rho u)}{\partial x} + \dfrac{\partial(\rho v)}{\partial y} = 0 \\ \rho c_p \left(u \dfrac{\partial T}{\partial x} + v \dfrac{\partial T}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\lambda \dfrac{\partial T}{\partial y}\right) + \mu \left(\dfrac{\partial u}{\partial y}\right)^2 \\ \rho T = \text{constant} \end{cases}$$

These equations were solved in closed form by Yen, by making use of Chapman's constant and noticing that $\mu\rho$ = constant. Therefore, as in the incompressible case, ψ, u and v can be expressed in finite form as the function of x and y, through the intermediary of a parameter ξ. Since the calculations are not essentially different from those of the previous case, they are not repeated here. They can be found in Reference 6.

Two numerical conditions were calculated for this case, corresponding to nozzle A of Reference 4 with $P_n/p_o = 2.53$ and $P_n/p_o = 2.1$. The results are plotted in Figure 8. One sees that there is general agreement between theoretical and experimental results. Any better agreement would seem suspicious, since the experimental data are not two-dimensional.

Turbulent, Incompressible, Two-Dimensional Annular Nozzle

The equations governing a two-dimensional turbulent incompressible jet are:

$$\begin{cases} u \dfrac{\partial u}{\partial x} + v \dfrac{\partial u}{\partial y} = \varepsilon(x) \dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

where $\varepsilon(x)$ is the kinematic eddy viscosity.

Solution in closed form was found by Goertler, (Reference 18). The solution may be written:

$$\psi = \frac{\sqrt{3}}{2} (K\sigma)^{1/2} x^{1/2} \tanh \eta$$

$$u = \frac{\sqrt{3}}{2} (K\sigma)^{1/2} \frac{1}{x^{1/2}} (1 - \tanh^2 \eta)$$

$$v = \frac{\sqrt{3}}{4} (K/\sigma)^{1/2} \frac{1}{x^{1/2}} (2\eta \operatorname{sech}^2 \eta - \tanh \eta)$$

and $\eta = \sigma \dfrac{y}{x}$

where $K = j/\rho$

σ = constant, for which Reichardt found experimentally the value 7.67.

A numerical application can therefore be made by the same mechanism as was used for the laminar incompressible case. It was made for the case of nozzle A of Reference 4 for the condition $P_n/p_o = 1.55$. The velocity distribution along the axis and on the dividing streamline were calculated as a function of x and are plotted in Figure 10. It is interesting to notice that the axial velocity decays much more rapidly in the turbulent case than in the laminar one. The same original velocity conditions are obtained at $x_o = 0.20$ ft. in the turbulent case against x = 664 ft. in the laminar one. In addition, for all practical purposes the velocity on the dividing streamline is nearly indistinguishable from the axial velocity.

The numerical solution here can only be approximate, since the base pressure depends upon the value of M_d at the beginning of the reattachment zone (zone II of Figure 4), which corresponds to an abscissa $\bar{x}_d$ which must be guessed at (for example as 75% of the abscissa of the stagnation point R of Figure 3). In this case, one finds a value $p_c/p' = 0.86$ which is in the same neighborhood as the value found in the laminar case.

It is dangerous to speculate on the worth of the theory presented above. It is a logical adaptation of the missile base pressure theories laboriously evolved over many years. Only tests on a two-dimensional configuration will make possible further progress.

Axially Symmetrical Case

For an axially symmetrical annular jet, the axes of reference and systems of coordinates are defined in Figure 11. One must carefully distinguish between the axis of revolution of the configuration and the "axis"of the jet.

A meridian section of the annular jet (Figure 11) looks exactly like a section of the two-dimensional jet (Figure 3). In particular, the conceptual scheme developed for the two-dimensional case can immediately be applied to the three-dimensional one. In view of the axial symmetry, one need only to consider a meridian section and then dividing stremline, stagnation point, and reattachment zone are formally the same in two and three dimensions. In particular, the formula giving the base pressure in two dimensions can still be applied if one uses for M_d the Mach number on the conical "dividing streamline" corresponding to the beginning of the reattachment zone.

The problem therefore becomes one of calculating the velocity distribution of the axially symmetrical annular jet. The mathematical formulation of this problem is similar to that for boundary-layer flow over a body of revolution whose radius $r_o(x)$ is large compared to the boundary-layer thickness.

In the laminar incompressible case, the equations of the jet are:

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\dfrac{\partial^2 u}{\partial y^2} \\ K u^2 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} \\ \dfrac{\partial (r u)}{\partial x} + \dfrac{\partial (r v)}{\partial y} = 0 \end{cases}$$

where K is the curvature of the axis of the jet (Figure 11) and r is the radius of any point inside the jet mixing region. The radius of the axis of the jet is r_0, and $r \approx r_0$. One will assume as in the two-dimensional case, for the purpose of calculating base pressures, that $K = 0$, and also disregard $\partial p/\partial x$. $r_0(x)$ is an unknown function which will have to be guessed at for a numerical solution.

The equations describing the flow reduce to:

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \nu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial (r_0 u)}{\partial x} + \dfrac{\partial (r_0 v)}{\partial y} = 0 \end{cases}$$

It is well known that this system of equations can be reduced to the corresponding system of two-dimensional equations by means of a Mangler transformation. Let us define new variables $\bar{x}$, $\bar{y}$ and $\bar{\Psi}$ which are functions of the original variables as follows:

$$\bar{x} = \int_0^x \frac{r_0^2(x)}{L^2}\,dx$$

$$\bar{y} = \frac{r_0(x)}{L}\,y$$

$$\bar{\Psi} = \frac{\Psi(x,y)}{L}$$

with $\partial\Psi/\partial y = r_0 u$ and $\partial\Psi/\partial x = -r_0 v$

$$\partial\bar{\Psi}/\partial\bar{y} = \bar{u} \qquad \partial\bar{\Psi}/\partial\bar{x} = -\bar{v}$$

by substitution, one finds for $\bar{x}$, $\bar{y}$, $\bar{u}$, $\bar{v}$ a system of equations which is identical to that of two-dimensional flow.

For numerical calculations, it is necessary to find an expression for $r_0(x)$. As a first approximation, one can take a linear relation

$$r_0 = R_0 - k\,x$$

In the numerical example mentioned before, $R_0 = 0.19$ ft. and $k = 0.5$,

$$r_0 = 0.19 - 0.5\,x$$

Further, it can be shown that one should take $L = R_0$. If one wanted to represent the shape of the jet more exactly, one could use

$$\frac{r_0}{R_0} = 1 - \left(\frac{x}{a}\right)^2$$

If one considers the first approximation, the relationship between the coordinates may be written:

$$\bar{x} = x - 2.78\,x + 2.56\,x^2$$

$$\bar{y} = y\,(1 - 2.78\,y)$$

in addition, $\bar{u} = u$.

The formal expression of the velocity is the same, but it must be taken at different points in accordance with the formulas of transformation.

A numerical application to a laminar jet indicated that Mangler's transformation did not change appreciably the base pressure p_c/p'. The reason is that the velocity along the dividing streamline (Figure 7) is practically independent of x. However, this point requires further study and no firm conclusion can be reached now. It is important to find out experimentally whether two and three dimensional annular jet base pressures are nearly the same or not. However, it is not sure that the jets obtained experimentally will be laminar.

In the case of a turbulent jet, the equations describing the axially symmetric flow are:

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \mathcal{E}(x)\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial (r_0 u)}{\partial x} + \dfrac{\partial (r_0 v)}{\partial y} = 0 \end{cases}$$

Mangler's transformation can be applied as before is one postulates additionally that $\bar{\mathcal{E}}(\bar{x}) = \mathcal{E}(x)$. In this case the numerical results indicate a larger

value of p_c/p' for the three-dimensional case than for the two-dimensional one, because the three-dimensional u_d turns out to be smaller than the two-dimensional one. This seems to indicate an evolution of the base pressures in the right direction.

ANNULAR JETS CLOSE TO THE GROUND

Experimental Results

From the start, the development of a theory for the annular jet close to the ground was handicapped by the almost complete absence of experimental data. In particular, no data exist for the "two dimensional" equivalent of the annular jet which makes completely impossible the check of a two-dimensional theory if and when it is found. Furthermore, such data as do exist appear at first highly contradictory and can only be reconciled after careful study. A need clearly exists for a systematic coordinated program of experimental research if the behavior of annular jets is to be understood.

The only unclassified data concerning the ground effect on the thrust of annular jets which have been published to date represent the work of the NASA Lewis Research Center (Reference 4). In addition, the Langley and Ames Research Centers of the NASA have been investigating the phenomenon for some time and have collected considerable data which are as yet unpublished. Some of these data, References 19 and 20, although unpublished have been made available to interested parties for use and interpretation at their own discretion, pending analysis and publication by the NASA. A thorough discussion of available experimental data is made in Reference 6. It cannot be reproduced here because of its length and it will now be summarized briefly.

First of all, what is of concern is the variation of thrust augmentation with distance to the ground. Typical thrust augmentation curves are shown in Figure 12. One must be careful in comparing different tests to make sure that the same definition of thrust augmentation was used in all cases. In Figure 13, three possible definitions of the thrust augmentation are indicated. The corresponding curves are plotted in Figure 14 for nozzle A of Reference 4.

Enough data are available to indicate that in addition to the distance from the ground two factors play an important role: the nozzle pressure ratio and the aspect ratio. The aspect ratio is defined as the ratio of the mean circumference of the nozzle to its thickness. For a given aspect ratio and a given h/D (see notation on Figure 12) there is a pressure ratio which gives an optimum augmentation factor. For a given pressure ratio and a given h/D, the augmentation factor usually increases with aspect ratio for h/D smaller than one. The "critical altitude" defined in Reference 4 (altitude at which the thrust jumps discontinuously to its value outside the ground cushion) is most noticeable at low aspect ratios. As the aspect ratio increases, the discontinuity seems to smooth out and the corresponding altitude at which it occurs shifts towards the left, i.e., towards lower altitudes. Correspondingly, as the aspect ratio increases, the augmentation curve acquires a hump at an h/D around 0.4. At a high enough aspect ratio, the slope of the curve will actually reverse and become positive (Figure 12).

Figure 15 indicates that there is not a "universal" thrust augmentation curve as a simplified momentum theory discussed later would tend to indicate.

Figure 16 shows that there is a good correlation between all available data, if the aspect ratio is properly accounted for.

In all available tests, an important parameter is missing: the angle of the jet with the vertical. This angle was assumed zero at all times. It is hoped that more experimental data including this important variable will soon be available.

Non-Viscous Two-Dimensional Theory

A non-viscous elementary theory of the "two-dimensional" annular nozzle in proximity to the ground was developed independently in References 5 and 21. It can be found in both these references and in addition is discussed in References 2 and 6. Therefore, it will be treated very briefly here. Consider the configuration shown in Figure 17. Assume that the jet is thin, incompressible and inviscid. The jet sheet will be deflected as shown. Assume that the momentum of the jet is conserved downstream of the nozzle. By writing the equilibrium between centrifugal and pressure forces, one finds:

$$\Delta p = j/R$$

where j is the momentum discharged per second by unit length of the jet and R is the radius of curvature of the jet.

Then the augmentation factor is found to be:

$$A_j \approx 1 + \frac{1}{2\,h/b}$$

The formula assumes R = constant, as a consequence of Δp = constant (hydrostatic pressure within the cavity) and j = constant (momentum flux conserved along the jet).

This formula was extended by Chaplin (Reference 5) for the case where the jet makes an angle θ_o with the vertical. From simple geometrical relationships (Figure 18), one finds:

$$A_j = \cos\theta_o + \frac{1 - \sin\theta_o}{2\,h/b}$$

The formula was also extended by Chaplin to an axially-symmetrical nozzle close to the ground. The system of coordinates used in this case is shown in Figure 19. One still has:

$$\Delta p = j/R$$

where R is the radius of curvature of the jet axis and j is the momentum flux per unit width of the jet measured along a horizontal circumference.

If J is the total momentum, one has:

$$j = J/2\pi r$$

or: $$\Delta p = J/2\pi r R$$

In this equation Δp and J are constant. Therefore R varies like $1/r$. On the other hand, the intrinsic equation of the jet is:

$$\frac{d\theta}{ds} = \frac{1}{R} = \frac{2\pi r \Delta p}{J}$$

This equation defines the shape of the jet. After some manipulations, it can be solved and from there the thrust augmentation curve can be found.

The augmentation curve obtained from the axially symmetric solution is plotted in Figure 20, where it is compared with an approximate solution derived by applying directly the two-dimensional results.

If Figure 20 is compared with Figure 15 or 16, it can be seen that the momentum theory represents only qualitatively the experimental results inasmuch as it makes no provision for the changes due to variation of the aspect ratio.

It must be stated that, at this time, no theory can explain in Figures 15 and 16 the effects of the variation in aspect ratio, such as the hump noted at high aspect ratios. It is felt that the direction in which to go is towards a viscous analysis of the phenomena. Therefore, in the following paragraph, an approach towards the solution of the viscous equations of motion of an annular jet close to the ground is presented.

Note that, though all tests of Reference 4 were performed on nozzles of circular plan form, there is nothing magic about the circular planform and any plan form will give a thrust augmentation close to the ground.

Viscous Two-Dimensional Theory

It is necessary first to define a system of coordinates for a curved two-dimensional jet, then to find the viscous equation of the jet and its solution. This is done by analogy with the boundary-layer flow along a curved wall. In the latter case the choice of the system of coordinates is obvious (Figure 6). It is a curvilinear system of coordinates in which the x axis *is in the direction of the wall and the y axis normal to it. The stag*nation point is taken as the origin of coordinates. The curvilinear net therefore consists of curves parallel to the wall and of straight lines perpendicular to them. The corresponding velocity components are called u and v and the curvature at a point x is called K (x), which is a continuous function of x and is positive for convex and negative for concave curvature. The complete Navier-Stokes equations in such a system of coordinates were first derived by Tollmien in Reference 22. For easy reference, they can be found in References 6 or 23. There is no point in reproducing them here, since they are untractable.

If the curvature is moderately large, it is shown in Reference 23 that the equations of motion for the laminar boundary layer flow reduce to:

$$\left\{\begin{aligned} & u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \\ & \frac{\partial p}{\partial y} = \rho K u^2 \\ & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{aligned}\right.$$

The main difficulty in trying to apply these boundary layer equations to a curved jet lies in the definition of a system of coordinates. One is obliged to define an "axis", or a center line, of the curved jet. At the outset, the shape of the axis is not known, but one can assume that it can be obtained approximately by means of a non-viscous analysis. This axis can then be taken as the x-axis, the origin being obviously the nozzle of the jet. The y-axis is taken perpendicular to the x-axis. The x-axis then separates the jet as two regions, shown as I and II in Figure 6.

The boundary conditions for the half jet I will be on one side that its velocity be zero, and on the other side that it be equal to the velocity on the axis, which is a function of x only. The equations of motion for region I are the three above equations with K negative (concave curvature). The boundary conditions for the half jet II will be on one side that its velocity be equal to the velocity on the axis, and that on the other side it be zero. The equations of motion are the same as for region I, with K positive (convex curvature).

Assuming therefore that the equations of a viscous curved jet are the same as those of the laminar boundary flow along a curved jet shown above, the problem is to find a method of solution of these equations, the solution satisfying the boundary conditions of the curved two-dimensional jet.

One notices first that the second momentum equation is formally the same as the corresponding equation in the inviscid case. In the inviscid case, u = constant and $\nu = 0$. Therefore, the first and the third equation vanish, but the second one is unaffected. There is a difference between the two cases, i.e., that in the inviscid case u = constant and in the viscous one u is variable. However, in the viscous, one assumes that the total momentum flux across a section of the curved jet must be the same for all sections. Hence:

$$\rho\int_{-\infty}^{+\infty} u^2\, dy = \text{constant}$$

Since K is a function of x only, the pressure difference across the cavity, Δp, is given by:

$$\Delta p = K \rho \int_{-\infty}^{+\infty} u^2 \, dy$$

The integral of $\rho u^2 \, dy$ is the initial momentum flux, which is constant. Therefore, in the viscous case the expression of Δp is the same as that in the inviscid one. This explains the relative success of the simple momentum theory to represent the thrust augmentation of annular nozzles close to the ground.

Now, if one assumes, as is classically done for slot jets (Reference 1) that $\partial p/\partial x = 0$, then the two momentum equations become completely independant from each other. The system of equations is formally similar to the equations of a straight plane viscous jet to which is superposed the inviscid pressure difference Δp.

However, in the general case $\partial p/\partial x \neq 0$. Then the two momentum equations are linked together and cannot be solved separately. A solution can be found in principle if two assumptions are made: first, that the velocity profiles are similar, i.e., one introduces a parameter η such that:

$$\eta = \frac{1}{3 \nu^{1/2}} \frac{y}{x^{2/3}}$$

second, that the curvature of the jet assumes the form

$$K = \frac{A}{6 \nu^{1/2}} x^{-2/3}$$

This is a family of spiral curves and for the proper value of the constant A can approximate well the shape of the jet.

Assume a stream function of the form:

$$\Psi = \nu^{1/2} x^{1/3} f(\eta)$$

Obviously, $u = \partial \Psi/\partial y$, $v = -\partial \Psi/\partial x$

$$u = \frac{1}{3 x^{1/3}} f'(\eta)$$

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial \eta} \frac{\partial \eta}{\partial x}$$

$$\frac{\partial p}{\partial \eta} = \frac{\partial p}{\partial y} \frac{\partial y}{\partial \eta} = (K \rho u^2)(3 \nu^{1/2} x^{2/3})$$

hence:

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = +\frac{2}{9} K \nu^{1/2} f'^2 \frac{\eta}{x}$$

finally:

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{A}{27} \frac{\eta}{x^{2/3}} f'^2$$

one can calculate $\partial u/\partial x$, $\partial^2 u/\partial y^2$ and v in the usual manner in terms of f, η and x and, substituting the above results into the momentum equation, one finds:

$$f''' + f'^2 + f f'' + A \eta f'^2 = 0$$

if $A = 0$ (plane jet), the differential equation for f reduces to that of the plane jet problem.

This equation cannot be solved simply, but can be solved by means of a series development. The velocity distribution within the jet can be found as soon as f is known, which satisfies the proper boundary conditions.

CONCLUSION

This paper presents the results of a comprehensive effort to understand the flow mechanics of an annular nozzle both far from the ground and close to the ground. These results are reported at this time with the hope of stimulating further research, both theoretical and experimental in the field of annular jets which is rapidly becoming an important technological field. There is still a long way to go before the behavior of curved jets can be correctly understood and predicted.

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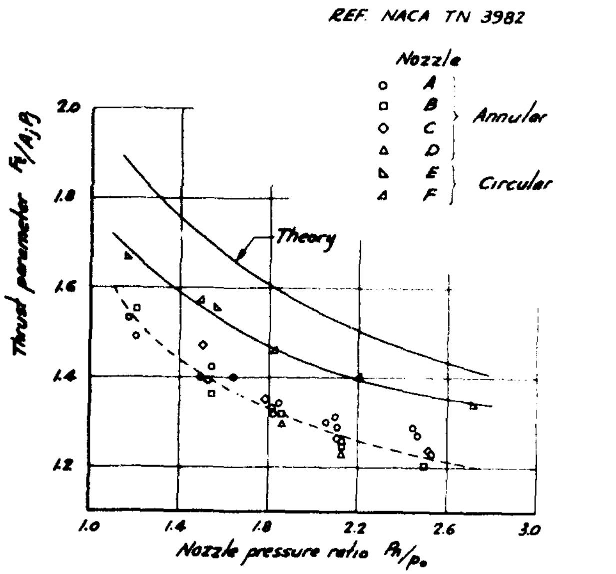

Figure 1. Characteristics of Annular and Circular Nozzles Far From the Ground

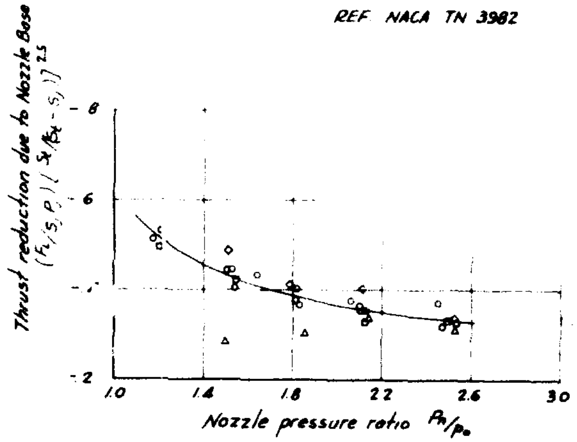

Figure 2. Thrust Loss Characteristics for Annular Nozzles Far From the Ground

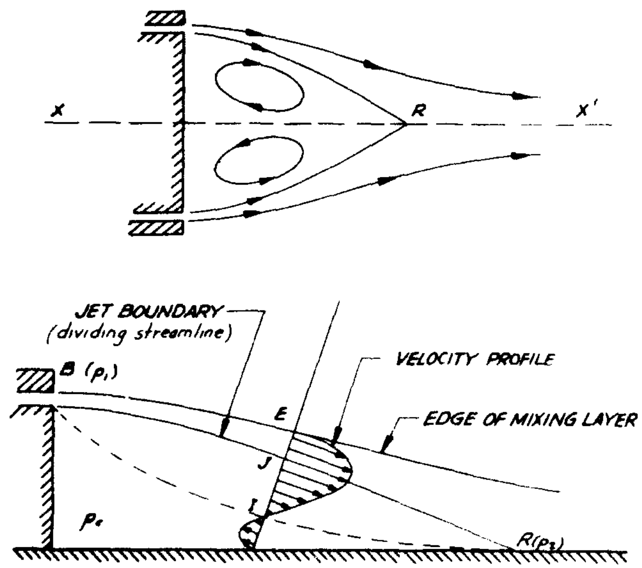

Figure 3. Velocity Distribution and Streamlines of a "Two-Dimensional" Annular Jet Far From the Ground

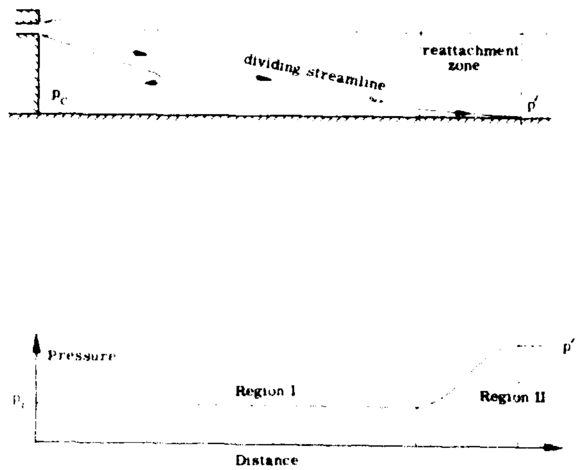

Figure 4. Pressure Distribution Along the Dividing Streamline of Figure 3

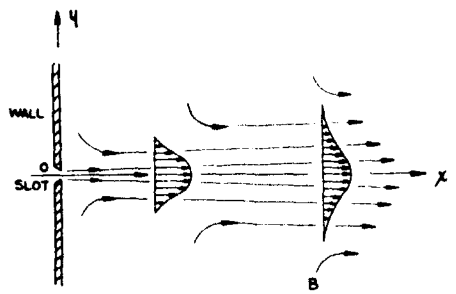

Figure 5 a. Velocity Distributions and Streamlines of a Jet Issuing From a Long Narrow Slot of a Wall

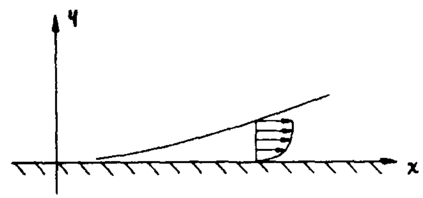

Figure 5 b. Typical Boundary Layer Profile Along a Flat Plate

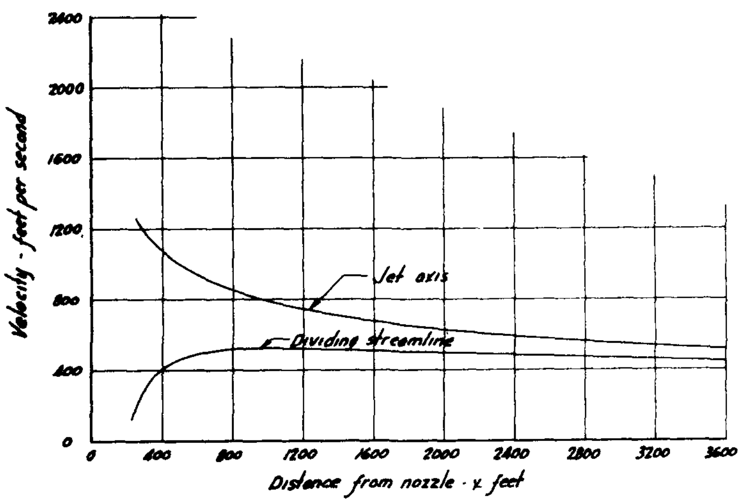

Figure 7. Calculated Velocity Along Axial and Dividing Streamlines (Laminar Jet)

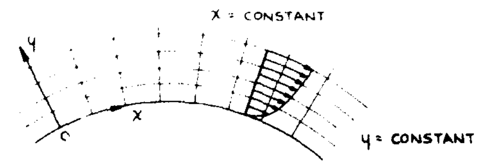

Figure 6 a. Coordinate System for Two-Dimensional Boundary Layer Flow Along a Curved Wall

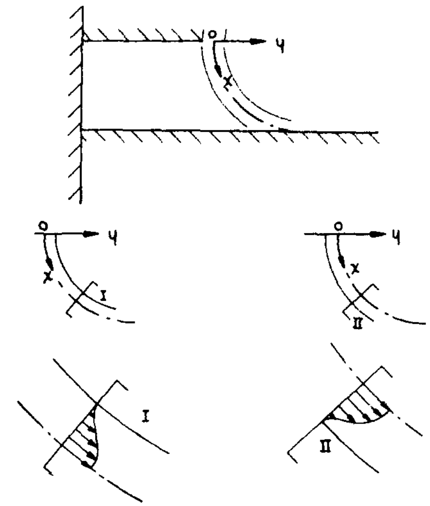

Figure 6 b. Coordinate System For a Curved Jet

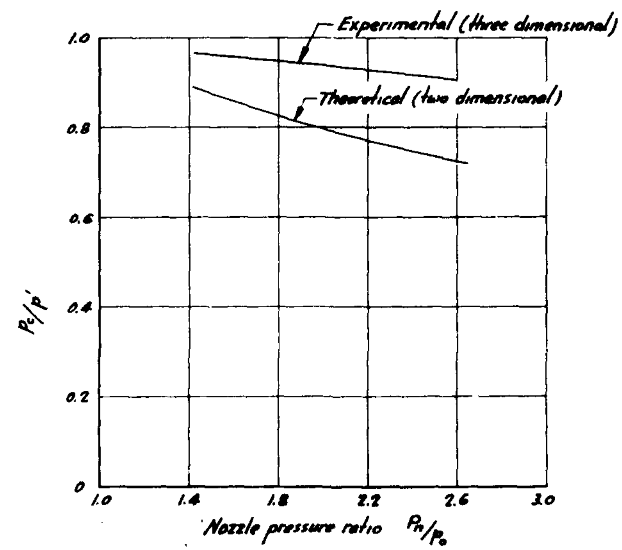

Figure 8. The Annular Jet Outside of Ground Effect: Comparison of Two-Dimensional Theory with Three-Dimensional Tests

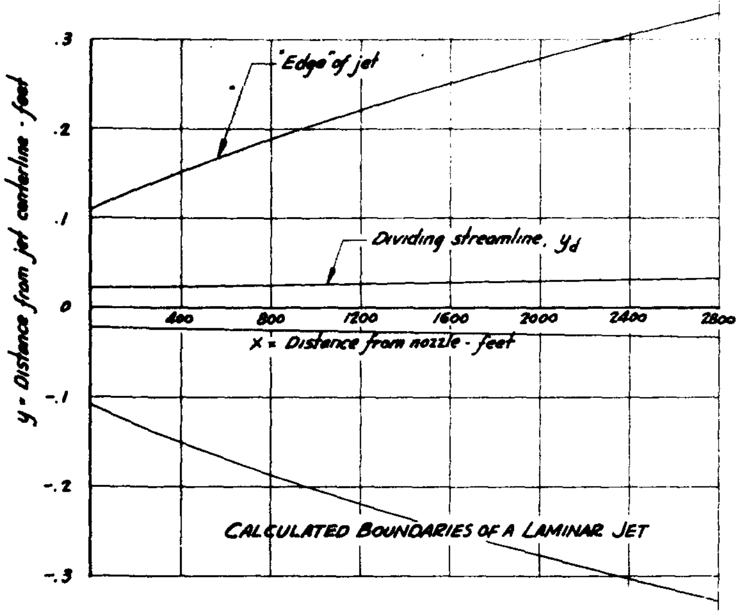

Figure 9. Calculated Boundaries of a Laminar Jet

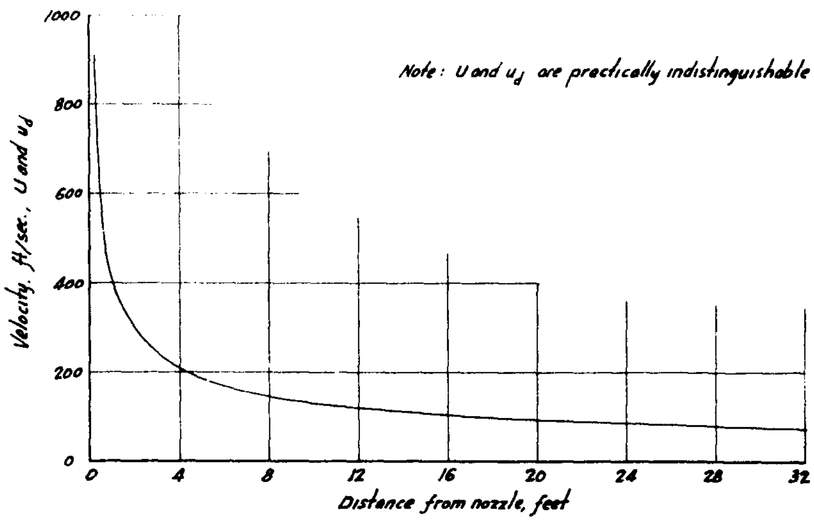

Figure 10. Calculated Velocity Along Axial and Dividing Streamline (Turbulent Jet)

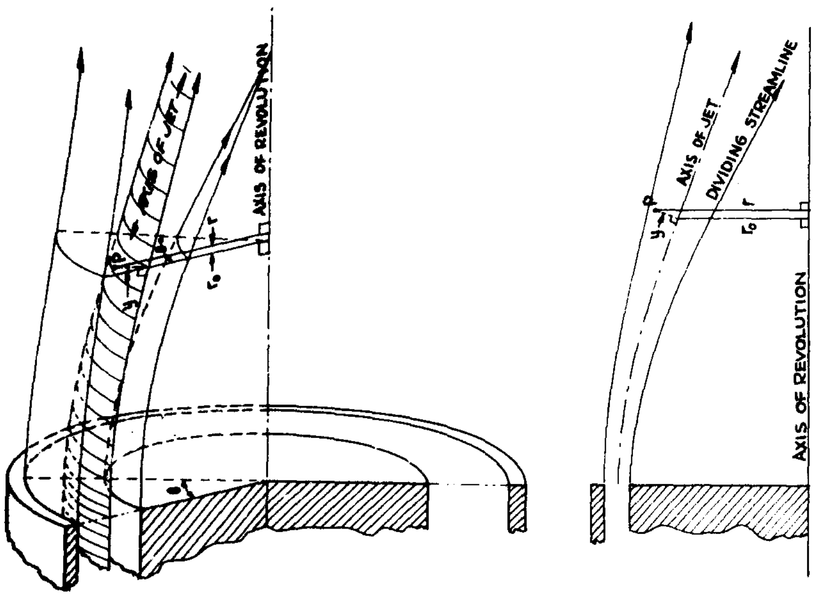

Figure 11. Schematic Diagram of Axially Symmetric Annular Nozzle Far From the Ground

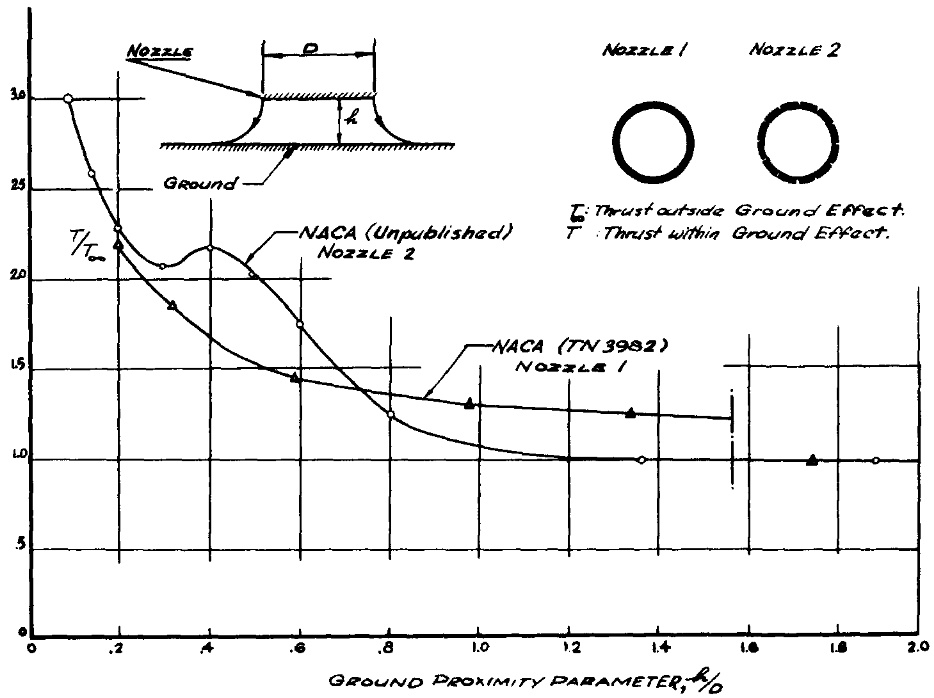

Figure 12. Thrust Augmentation of an Annular Nozzle in Proximity to the Ground

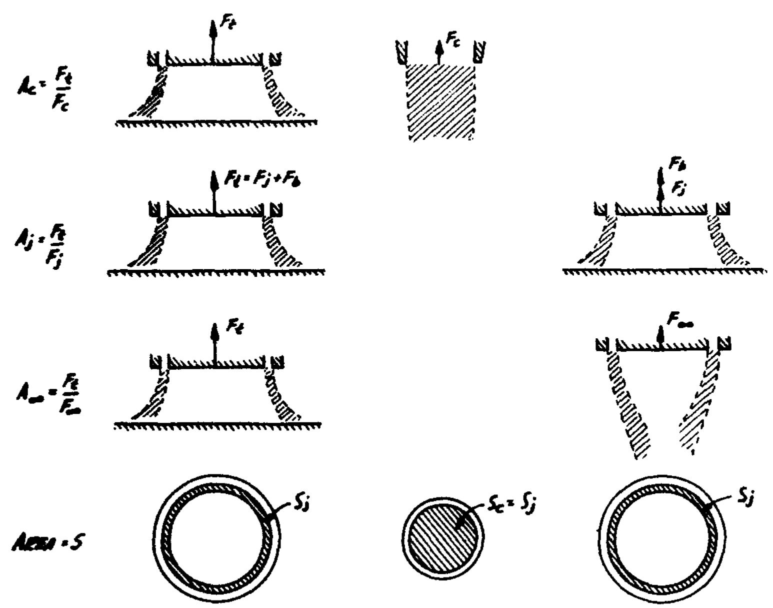

Figure 13. Definitions of Thrust Augmentation Factors for Annular Nozzles Close to the Ground

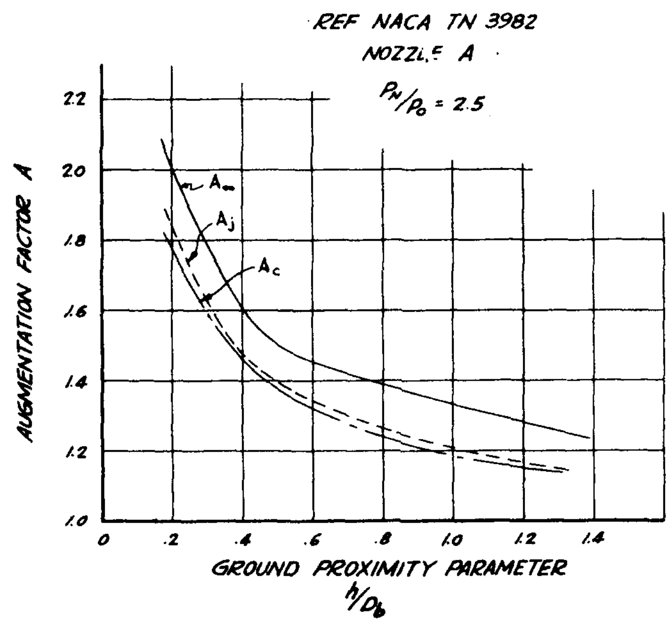

Figure 14. Augmentation Factors A_c, A_j, A_∞ versus Ground Proximity Parameter

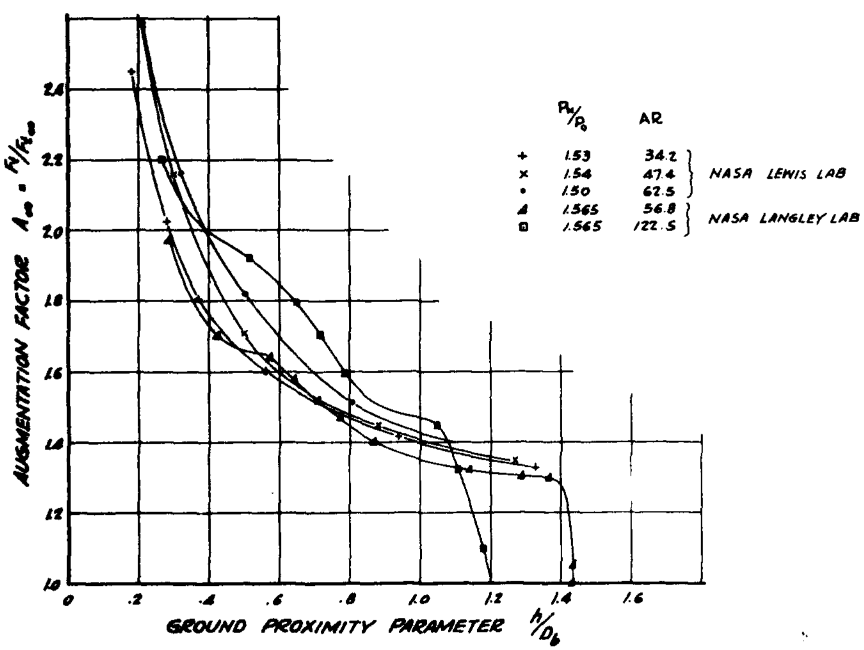

Figure 15. Typical Experimental Results Showing the Effect of Aspect Ratio

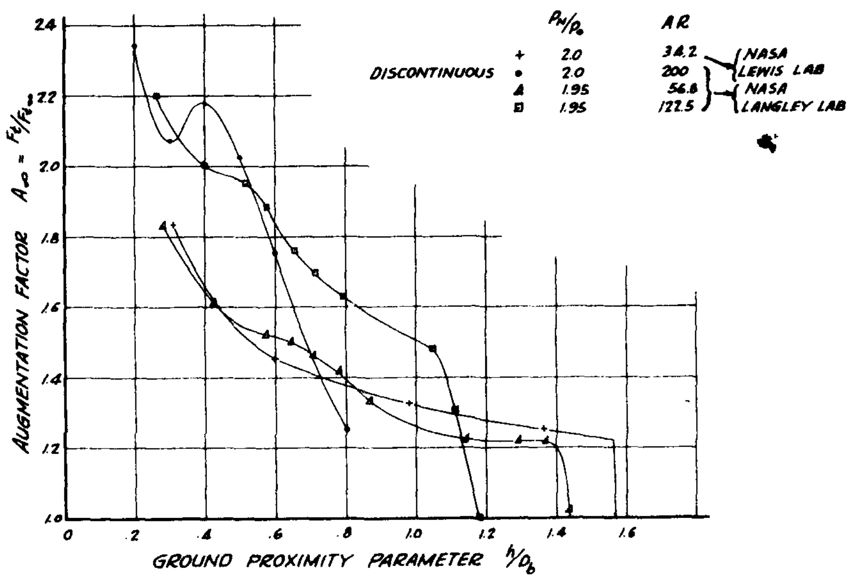

Figure 16. Comparison Between Experimental Data from NASA Lewis and Langley Laboratories

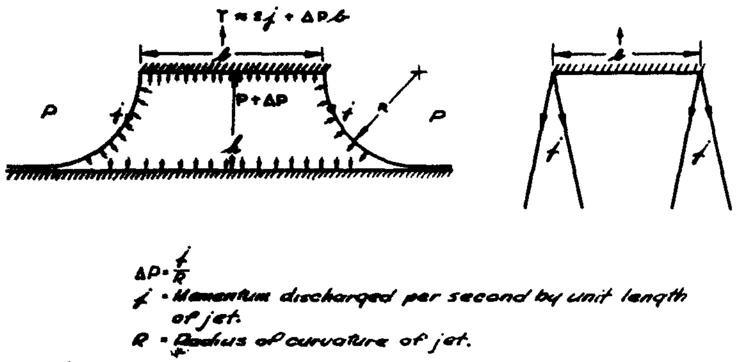

Figure 17. Approximate Expression for the Thrust Augmentation of a Two-Dimensional Slot-Jet Nozzle in Proximity to the Ground

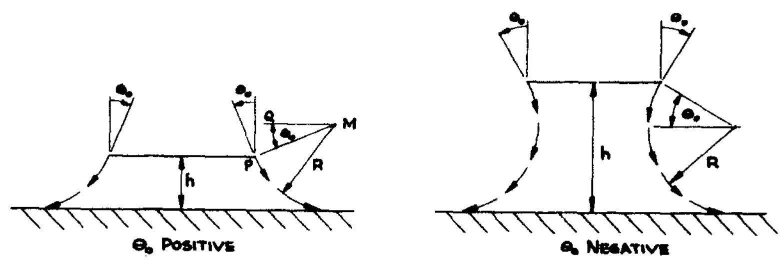

Figure 18. Two-Dimensional Annular Nozzle Close to Ground (Jet Inclined with Respect to Base)

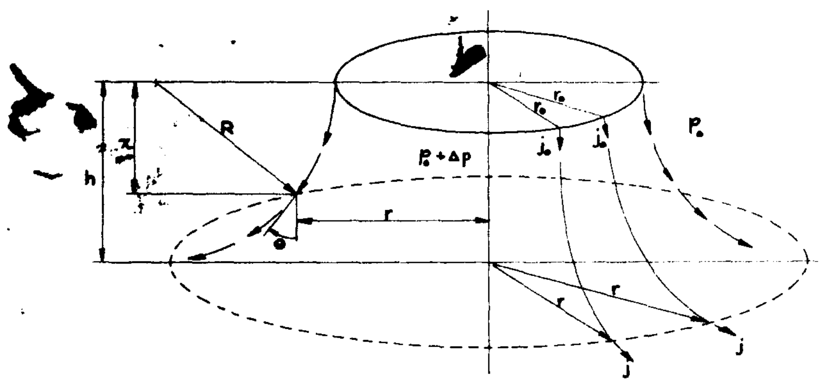

Figure 19. Schematic of Three-Dimensional Annular Jet Close to Ground

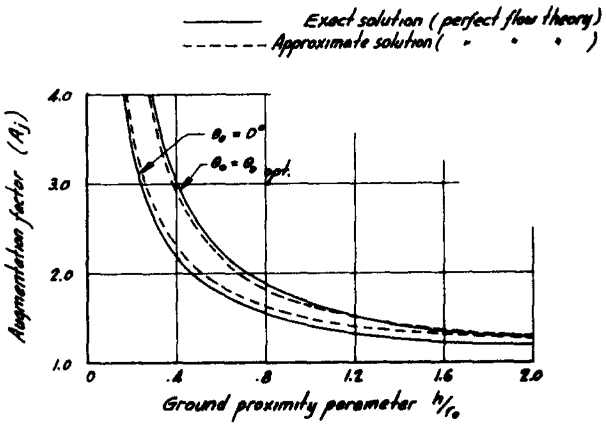

Figure 20. Comparison Between Inviscid Exact and Approximate Theories for an Axially Symmetric Jet